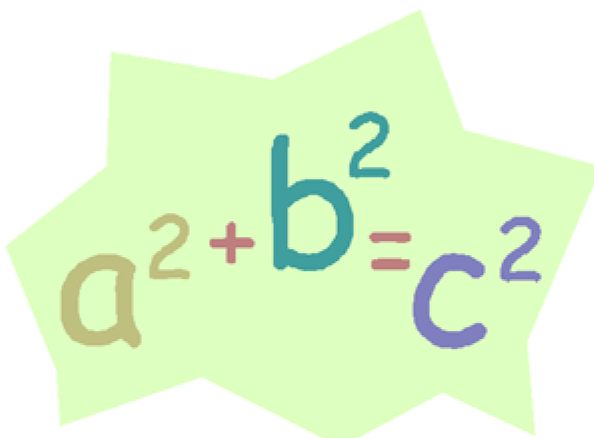
Developing
Algebraic Thinking



DEVELOPING ALGEBRAIC THINKING

Algebra is an important branch of mathematics, both historically and presently.

"… algebra has been too often misunderstood and misrepresented as an abstract and difficult subject to be taught only to a subset of … students who aspire to study advanced mathematics; in truth, algebra and algebraic thinking are fundamental to the basic education of all students… Algebra is frequently described as 'generalized arithmetic', and indeed, algebraic thinking is a natural extension of arithmetical thinking.

To think algebraically, one must be able to understand patterns, relations and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand quantitative relationships; and analyze change in various contexts.

In high schools, students create and use tables, symbols, graphs and verbal representations to generalize and analyze patterns, relations and functions with increasing sophistication and they flexibly convert among these various representations. They compare and contrast situations modeled by different types of functions and they develop an understanding of classes of functions, both linear and non-linear, and of their properties.

High school students continue to develop fluency with mathematical symbols and become proficient in operating on algebraic expressions in solving problems. Their facility with representations expands to include equations, inequalities, systems of equations, graphs, matrices and functions, and they recognize and describe the advantages and disadvantages of various representations for each particular situation. Such facility with symbols and alternative representations enables them to analyze a mathematical situation, choose an appropriate model, select an appropriate solution method and evaluate the plausibility of their solutions.

High school students develop skill in identifying essential quantitative relationships in a situation and in determining the type of function with which to model the relationship. They use symbolic expressions to represent relationships arising from various contexts, including situations in which they generate and use data. Using their models, students conjecture about relationships and formulae, test hypotheses and draw conclusions about the situations being modeled." (Greenes, *et. al.*, 2001, pp. 1-4)

Algebra allows its user to go from a conjecture to a level of certainty. For example, try these multiplications:

$$7 \times 9 = 63$$
$$13 \times 15 = 195$$
$$99 \times 101 = 9999$$

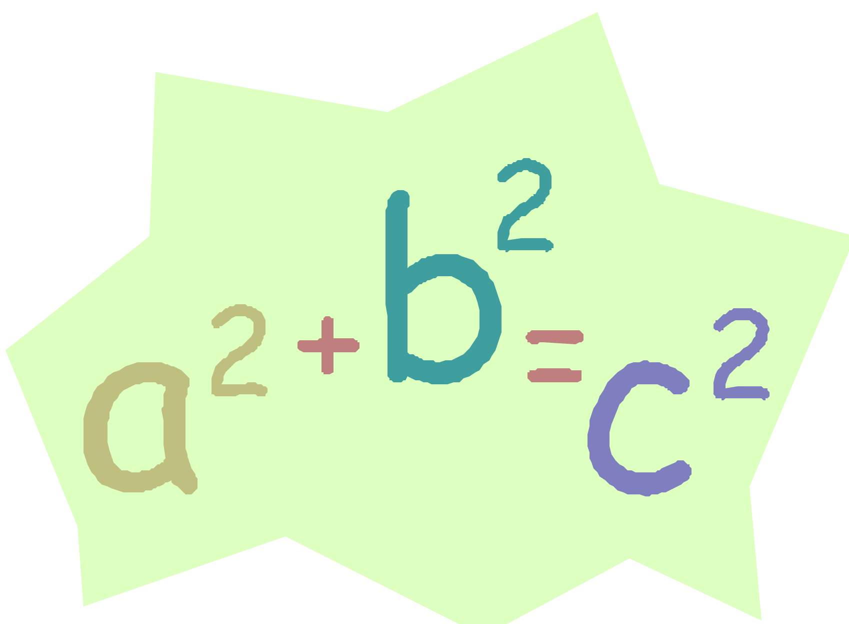
Notice that:

$$63 = 8^2 - 1$$
$$195 = 14^2 - 1$$
$$9999 = 100^2 - 1$$

It seems that if you multiply two numbers that are 2 apart, the answer is 1 less than the square of the number in between. But you can't be sure if this is always true. Algebra provides the tools that allow you to show that is it always true.

Two numbers that are 2 apart could be called x and $x + 2$. If you multiply these together, you get $x(x + 2) = x^2 + 2x$. The number in between x and $x + 2$ would be $x + 1$. Squaring $x + 1$, you get $(x + 1)^2 = x^2 + 2x + 1$.

So, if you subtract 1 from $(x + 1)^2$, you get $x^2 + 2x$, which can be factored as $x(x + 2)$, which is the same result as above. Since $x + 1$ is in between x and $x + 2$, you can say that for any value at all, it is true that if you multiply two numbers that are 2 apart, the answer is found by subtracting 1 from the square of the number in between.



REPRESENTING EXPRESSIONS USING SYMBOLS

One of the stumbling blocks in algebra is how to represent expressions using symbols. Below, you can relate words to symbols as well as visuals to symbols.

Words to Symbols

VERBAL EXPRESSION	CORRESPONDING SYMBOLS AND EXAMPLES
a number	**x (or any other letter or symbol)**
one more than a number	**$x + 1$**
twice a number	**$2x$**
two consecutive numbers	**x and $x + 1$, or $a – 1$ and a**
numbers that are 4 apart	**x and $x + 4$, or a and $a – 4$, or $b – 2$ and $b + 2$**

Visuals to Symbols

Materials called algebra tiles can be used when representing symbolic expressions. They are concrete and should have two sides, either coloured or marked differently.

Piece 1:

Side 1 **Side 2** **OR** **Side 1** **Side 2**

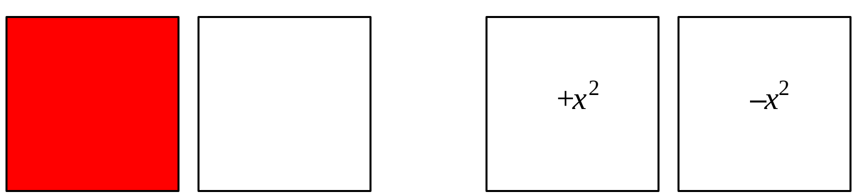
Side 1: Dark (+1)
Side 2: White (–1)

Piece 2:

Side 1 **Side 2** **OR** **Side 1** **Side 2**

Side 1: Dark (+x)
Side 2: White (–x)

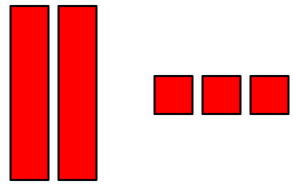
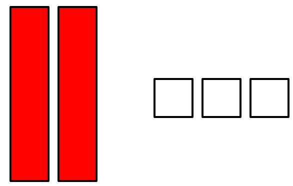
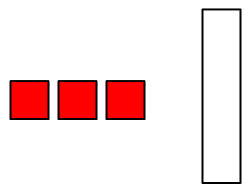
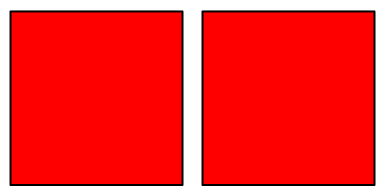
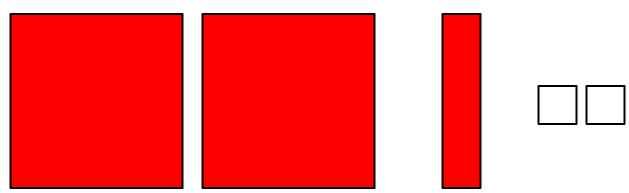
Piece 3:



Side 1 **Side 2** **OR** **Side 1** **Side 2**

Side 1: Dark (+x^2)
Side 2: White (–x^2)

Various Symbolic Representations

$2x + 3$	
$2x - 3$	
$3 - x$	
$2x^2$	
$2x^2 + x - 2$	

If it is necessary to use two variables, for example x and y, a set of pieces like those shown above can be used, but the length of the y-piece and the length and width of the y^2-piece should be different from the x-pieces. You may find it useful to work with similar materials with integers first. The following illustrates how to work with these materials.

Integers

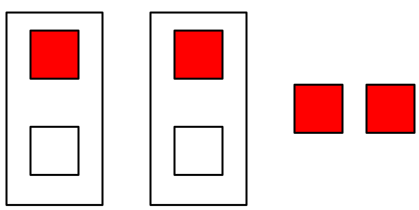
Before algebra, learners should have ample opportunity to work with integers. Money, temperature change and change in water levels are real life situations that can be used to investigate the concept of integers. To give learners a hands-on experience with integers, algebra tiles can be used. Integers are signed numbers, numbers with a sign that indicates direction. So, for example, 3 means the same as +3. Negative numbers are always shown with a minus sign.

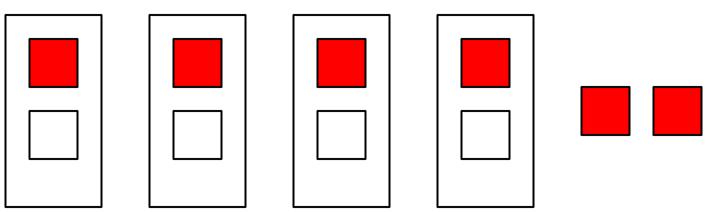
As a substitute, if tiles with different colors on different sides are not available, cut out squares of cardboard or mark squares of plastic, one side marked with +1 and the other side marked with –1.

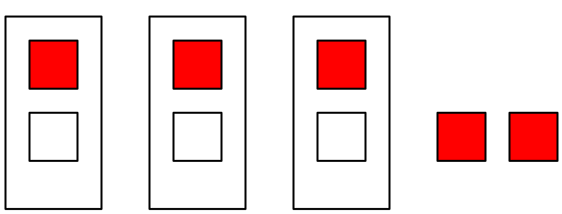
Much of the work with integers is dependent on the understanding that $+1 + (-1) = 0$ or, more generally, $(+x) + (-x) = 0$. Although this cannot be explained other than indicating that this is what defines $(-x)$, it may be helpful to refer to a real referent such as: if the temperature is at 1^{o}C and it goes down 1^{o}C, it is now 0^{o}C, i.e. $+1 + (-1) = 0$. This can be applied when working with tiles, as is demonstrated below.

The Zero Principle

Using an equal number of red and white or +1 and –1 tiles is a way to represent zero. In the examples below, the zero principle is used to maintain the integer +2.

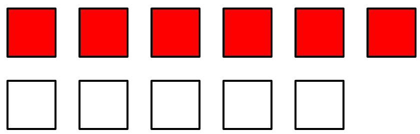






Introductory Exercise

1. Using unit tiles, have learners shake their tiles in their hands and release them onto their desks.
2. Have the learners arrange the tiles according to colour.
3. Encourage the learners to write down the difference between the number of red tiles and the number of white tiles.
4. Describe the same situation as an integer.



In this example, there is one more red tile than white tile, which you write as +1.

Now Try This!

1. Working in groups of two or more learners, take a handful of red and white tiles and place them in a cup. Shake them and dump them on the desk.
2. Have the learners arrange the tiles and write the difference between the numbers as integers.
3. After 10 attempts, have learners arrange the integers from least to greatest.

Also Try This!

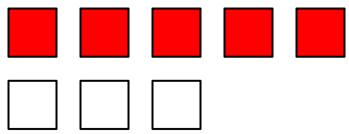
Have learners create 5 different models for each of the following integers.

(a) –1 **(b)** +2 **(c)** –4 **(d)** 0

ADDING AND SUBTRACTING INTEGERS

Adding

Use the algebra tiles to model (+5) + (–3)



You can remove equal numbers of tiles without changing the integer. That is, one red tile and one white tile represent zero. So, if three red tiles and three white tiles are removed, there is no change in value, but it is easier to see that the value is 2 red tiles or +2.

Therefore, (+5) + (–3) = +2.

Use red and white tiles to add each pair of integers.

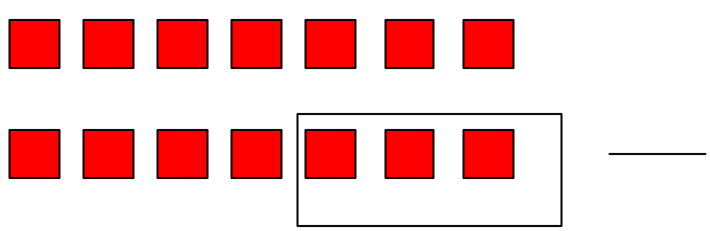
(+6) + (–4)	(+3) + (–6)	(–4) + (–3)
(-4) + (+1)	(–2) + (–6)	(+3) + (–3)

Subtracting

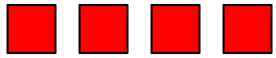
When adding integers, you combine groups of tiles. When subtracting, you do the opposite; you remove tiles from a group of tiles. You can illustrate this using the following examples.

1. (+7) – (+3)

   To model this problem, first show 7 red tiles, and then take away 3 red tiles. Tell what is left. Most students can do this mentally. It can also be shown with tiles:



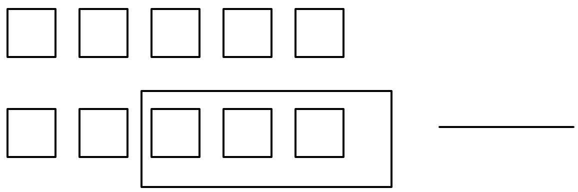
This results in the following diagram:



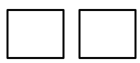
When you are subtracting 3 tiles you are subtracting +3. Therefore (+7) – (+3) = +4.

2. (–5) – (–3)

To model this problem, first show 5 white tiles, and then take away 3 white tiles. Tell what is left. Most students can do this mentally. It can also be shown with tiles:



This results in the following diagram:



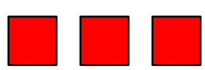
When you are removing the three white tiles, you are subtracting –3. Therefore, (–5) – (–3) = –2.

Notice how much easier it is to think 5 whites take away 3 white is 2 whites than learning complicating rules about changing signs.

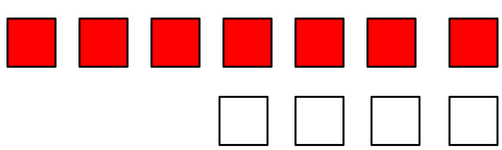
3. (+3) – (+7)

To model this problem, first show 3 red tiles, and then you are to take away 7 red tiles. Oh no! There are not 7 red tiles to take away.

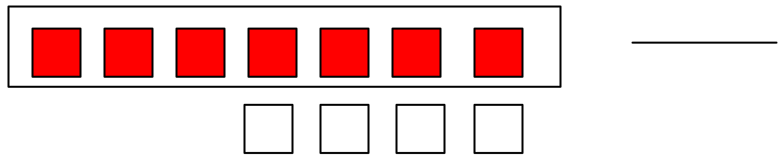
You would start by placing three red tiles on the desk.



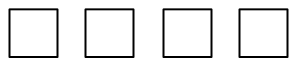
Since 7 tiles are not available to subtract from three, you must add a form of zero that has 7 red tiles. Since you already have 3 red tiles, you need to add 4 more red tiles to the existing 3 tiles. For every tile that you add to the existing integer, you must add the opposite tile so that you do not change the value of the integer.



By adding zero, you haven't changed the integer that the red tiles represent. Now you can subtract or remove the 7 red tiles leaving you with 4 white tiles. You write (+3) – (+7) = –4,

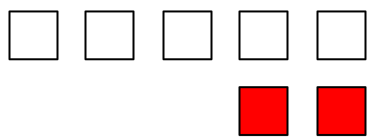


which becomes:

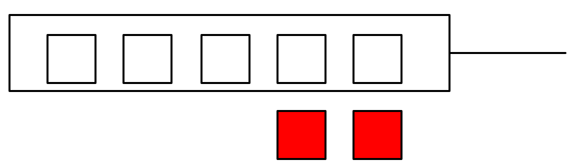


4. (–3) – (–5)

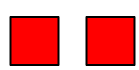
To model this problem, first show 3 white tiles, with the intention of taking away 5 white tiles. Again, you need to add some form of zero so that you have enough tiles to subtract.



You can remove the white tiles leaving 2 red tiles,



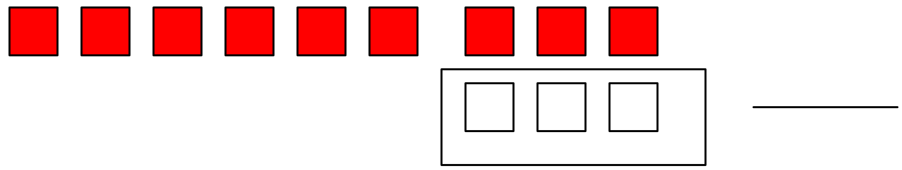
which becomes:



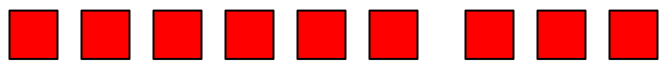
You can write (–3) – (–5) = +2.

4. (+6) – (–3)

To model this problem, first show 6 red tiles, and then add 3 red and 3 white tiles, which has the effect of adding zero.



Remove the 3 white tiles, leaving 9 red tiles. You can write this as (+6) – (–3) = +9. Therefore, it makes sense that (+6) – (–3) = (+6) + (+3). You had 6 red and had to add 3 red.



Try These!

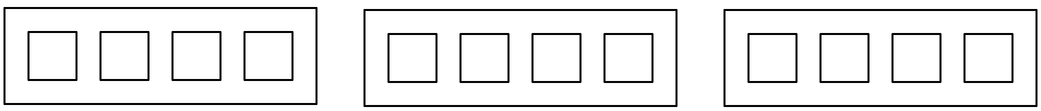
(+6) – (+2)	(–4) – (–3)	(+4) – (–2)
(–5) – (+3)	(–3) – (–4)	(+3) – (+7)

MULTIPLYING AND DIVIDING INTEGERS

Multiplying

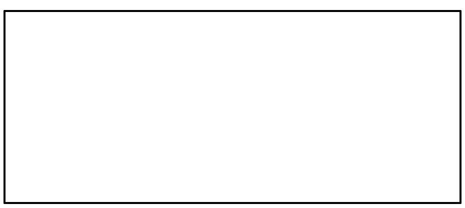
Multiplication of integers is directly related to whole number multiplication and division. Whole number multiplication can be seen as repeated addition. The first factor tells how many sets there are or how many are added, beginning with zero. The second factor tells what is in the set.

Integer multiplication with a positive first factor can be thought of in the same way. For example, $(+3) \times (-4)$ is seen as three sets of negative 4.

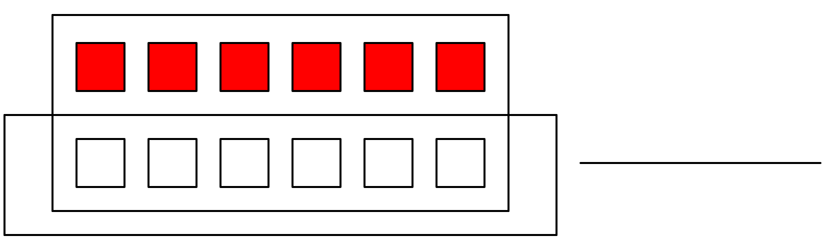


But $(-2) \times (-3)$ is more problematic. What would –2 copies of a set mean? What can be done is to think of $(-2) \times (-3)$ as $0 - [2 \times (-3)]$, which is two sets of –3 less than 0.

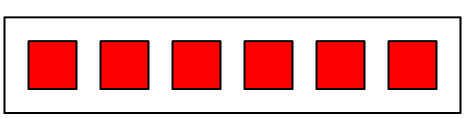
Begin with zero.



You want to remove 2 sets of –3, so you need to find a way to model zero that includes the 6 white tiles you will want to remove.



After you remove 2 sets of –3, or 6 white tiles, you are left with the following.



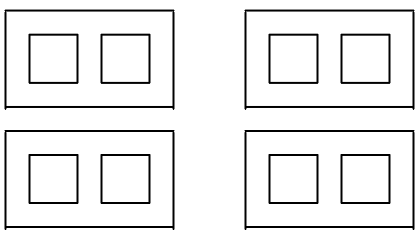
Therefore, (–2) × (–3) = +6.

The rule of “like signs yields positive products” and “unlike signs yields negative products” becomes very apparent using this model.

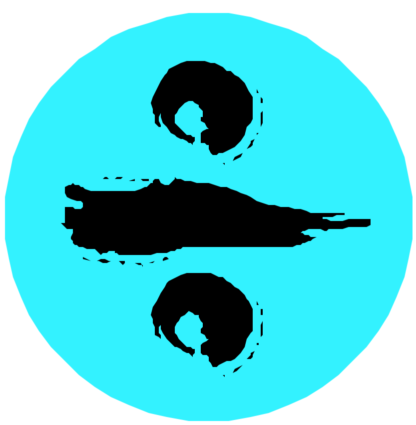
Dividing

Division of integers is also related to the concept of whole number division. For example, 6 ÷ 2 = 3 could be seen as what you do to find out how many sets of 2 make 6, or it could be asking for the size of the set if 2 of them are needed to make 6.

For example, consider the problem –8 ÷ (–2). It is easy to show that –8 can be divided up into 4 groups of –2. So, –8 ÷ (–2) = +4.



But how would you solve the problem –8 ÷ (+2)? This time, it might be better to think about what you would multiply +2 by in order to get –8. Since you know that (–4) × (+2) = –8, it follows that –8 ÷ (+2) = –4.



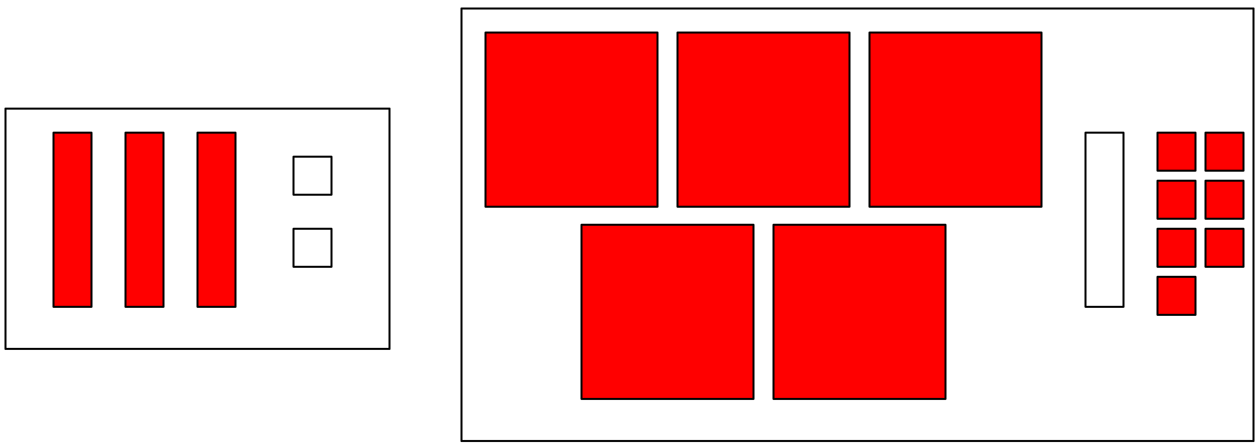
OPERATIONS WITH ALGEBRAIC EXPRESSIONS

You can use similar approaches to model operations involving algebraic expressions by using the algebra tiles that were introduced earlier.

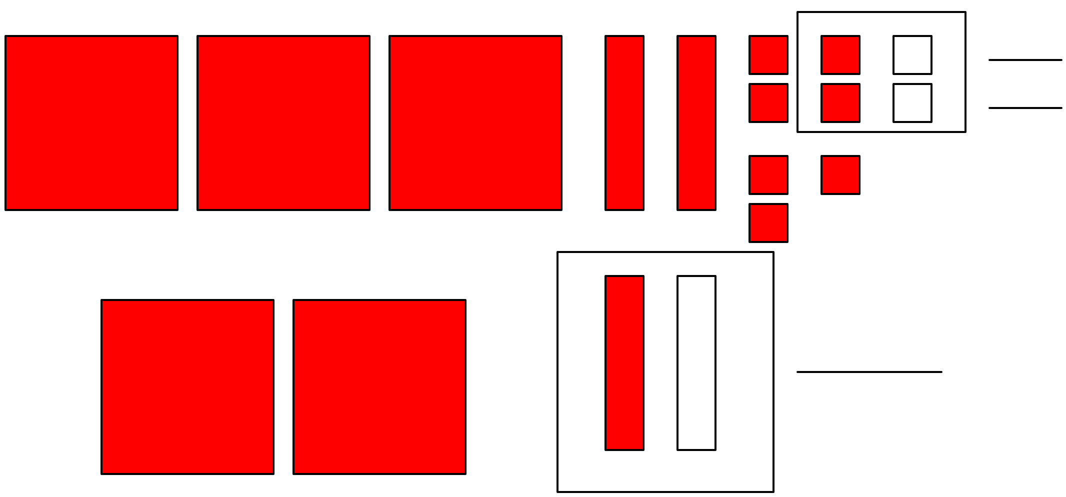
Adding

$(3x - 2) + (5x^2 - x + 7)$

First, you can model both polynomials.



Combining gives:

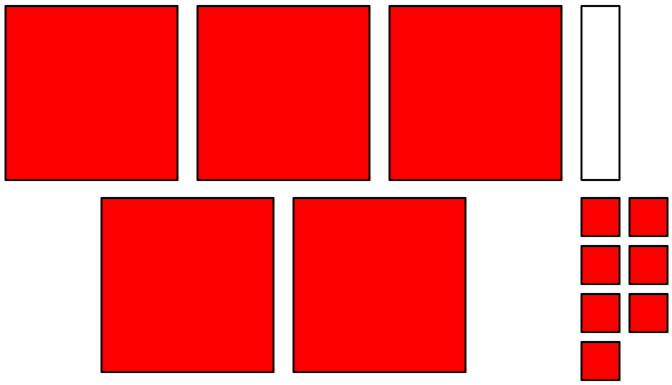


Removing equal numbers of opposite tiles leaves you with an answer of $5x^2 + 2x + 5$.

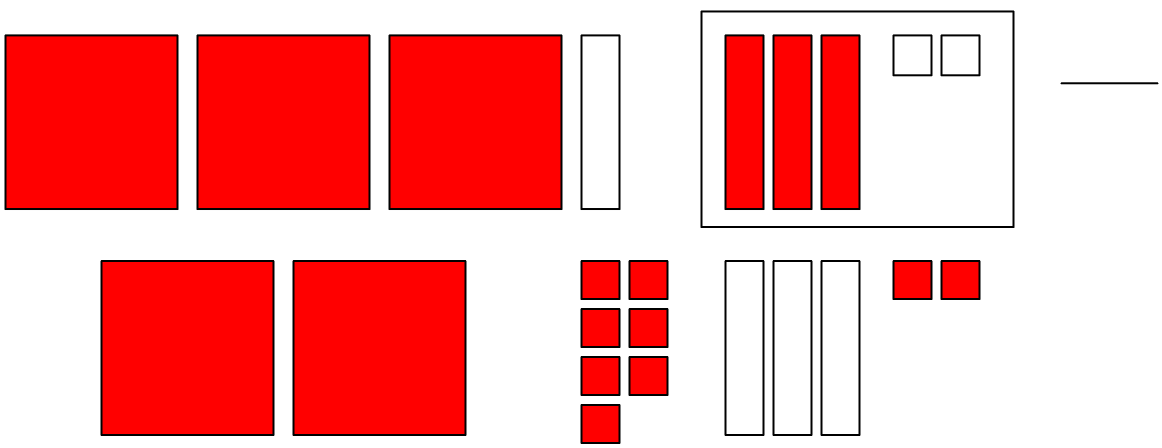
Subtracting

$(5x^2 - x + 7) - (3x - 2)$

Begin by modelling $5x^2 - x + 7$.



You have to remove $3x - 2$, but you do not have enough of the proper tiles to remove. Therefore, you will add enough tiles using the zero principle so that you can remove the required number of tiles.



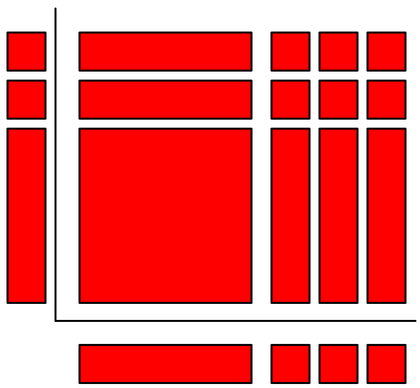
This leaves you with $5x^2 - 4x + 9$.

Multiplying

To multiply two algebraic expressions using algebra tiles, you may be able to use the fact that the product of two factors can always be represented using a rectangular array. Therefore, to find the product, you create a frame having each factor as a dimension. Then you can fill the resulting rectangle with the appropriate tiles, as demonstrated in the examples that follow. It is important to remember how signed numbers are multiplied together when filling in the array with the appropriately coloured tiles.

1. $(x + 2)(x + 3)$

$x + 2$

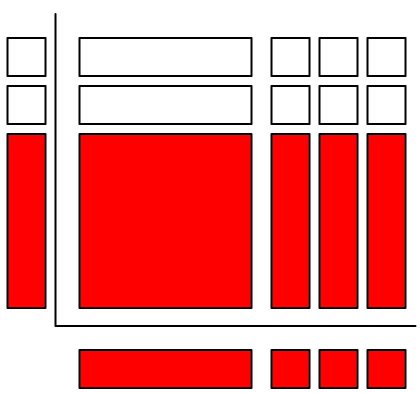


$x + 3$

By looking at the rectangular array, you see that the answer is $x^2 + 5x + 6$.

2. $(x - 2)(x + 3)$

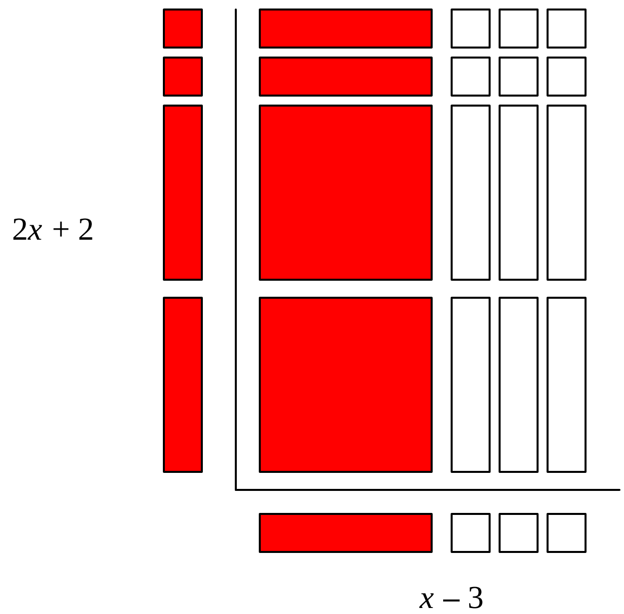
$x - 2$



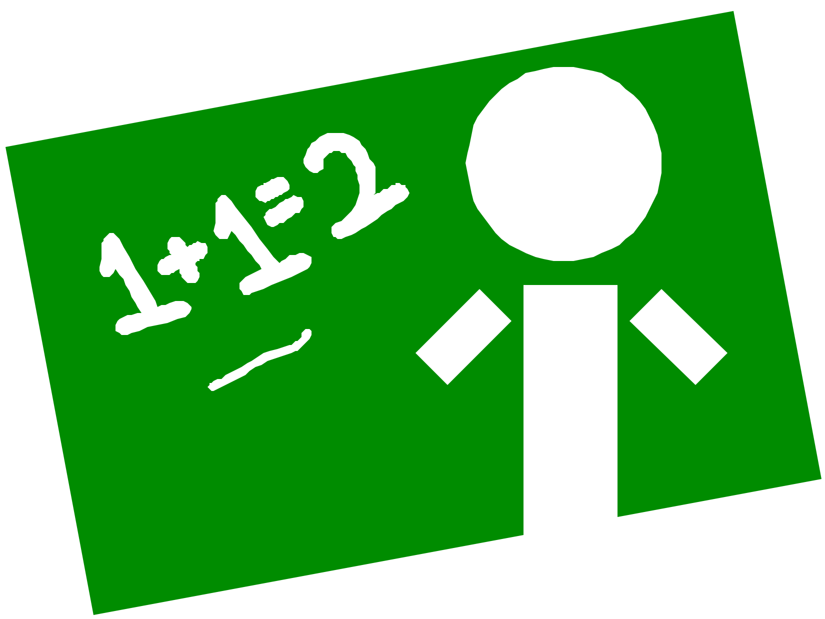
$x + 3$

Remember that a positive times a negative is a negative, resulting in the white tiles in the array. By looking at the rectangular array, you see that the answer is $x^2 + 3x - 2x - 6 = x^2 + x - 6$.

3. $(2x + 2)(x - 3)$



Again, remember that a positive times a negative is a negative, resulting in the white tiles in the array. By looking at the rectangular array, you see that the answer is $2x^2 + 2x - 6x - 6 = 2x^2 - 4x - 6$.



TRY THIS

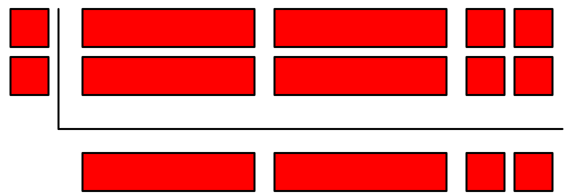
Multiply each pair of binomials, using the algebra tiles to help you. Which ones form squares? How could you have predicted this?

BINOMIALS	WORK	PRODUCT
$(x + 1)(x + 5)$		$x^2 + 6x + 5$
$(x + 2)(x + 2)$		
$(x - 4)(x + 7)$		
$(x + 8)(2x + 3)$		
$(x - 1)(2x - 1)$		
$(2x - 2)(x - 5)$		
$(x - 4)(3x - 7)$		
$(x - 3)(x - 3)$		
$(3x + 1)(2x - 5)$		
$(2x + 2)(4x - 5)$		

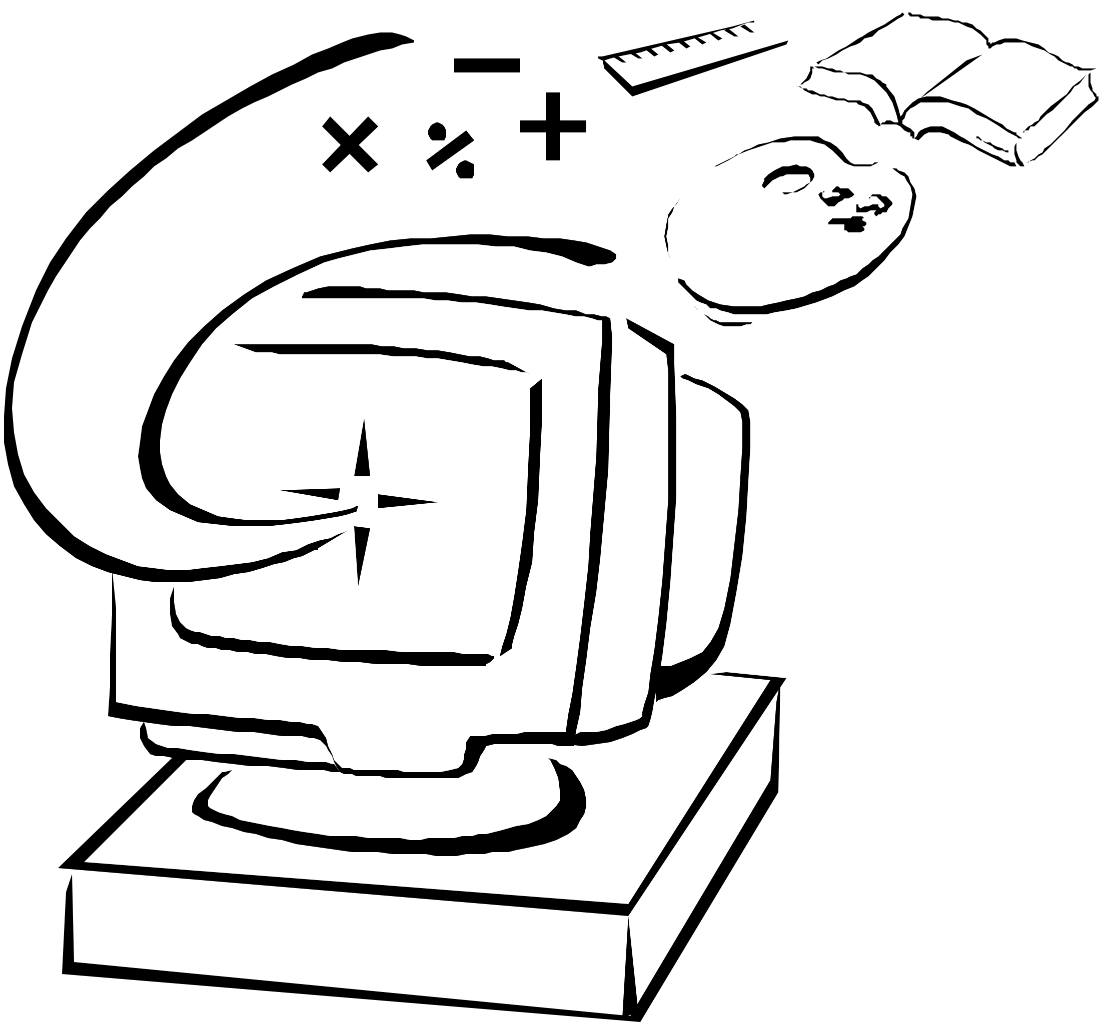
Dividing

Since division is the inverse operation of multiplication, you can use the same process as for multiplication, but in reverse. You begin by constructing a rectangular array from the algebra tiles that make up the first expression in the division problem and having the second term as one of the dimensions of the rectangle. The answer to the problem will be the tiles that make up the other dimension.

1. $(4x + 4) \div 2$

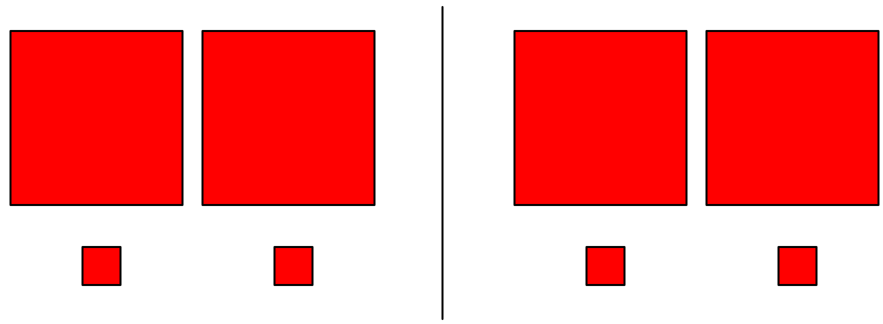


Therefore, the answer is $2x + 2$.



When you are dividing by an integer, another approach is to try and arrange the tiles into groups with the same number and type of tiles in each group.

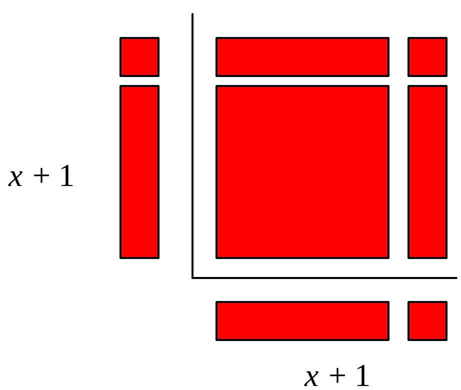
2. $(4x^2 + 4) \div 2$



You can see that there are 2 groups, each containing $2x^2 + 2$, which is the answer to the problem.

3. $(x^2 + 2x + 1) \div (x + 1)$

In this case, the first strategy works best, as $x + 1$ is not an integer.



From the diagram, you can see that the answer is $x + 1$.

SOLVING EQUATIONS

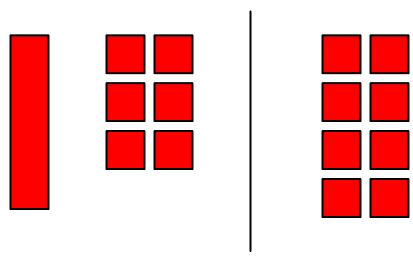
It is often necessary to solve equations when working with algebraic expressions. The process may be used to find the value of a variable that makes a certain equality true or it may be used to verify that two expressions are always equivalent. There are many approaches that can be used when solving linear equations. Several models are shown below.

Using Algebra Tiles

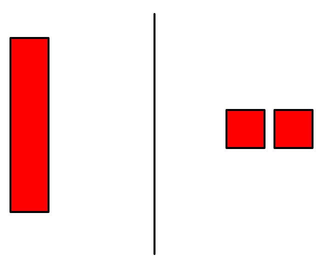
When solving algebraic equations by modeling them with algebra tiles, it is important to remember the basic principle underlying this process. The same number of tiles can be added or subtracted from each side of an equality without disturbing the equivalence. This principle is called the balance principle. The objective is to only have x on one side of the balance and a number, which will be equivalent to x, on the other side.

Ex. 1: Solve $x + 6 = 8$

Step 1: Model $x + 6$ on one side of the balance and +8 on the other side.



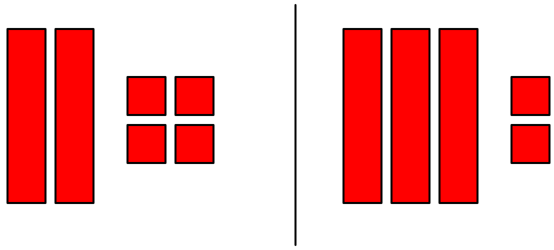
Step 2: Remove 6 red (or six +1) tiles from both sides.



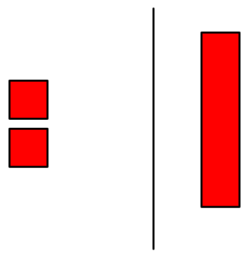
Step 3: Notice that what remains is $x = 2$.

Ex. 2: Solve $2x + 4 = 3x + 2$.

Step 1: Model $2x + 4$ on the left and $3x + 2$ on the right.



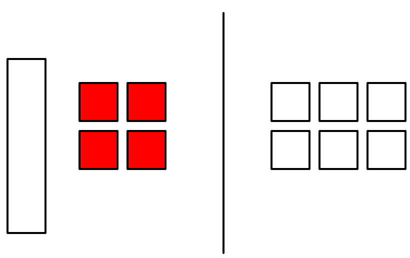
Step 2: Remove 2 of the $+x$ and 2 of the $+1$ from each side.



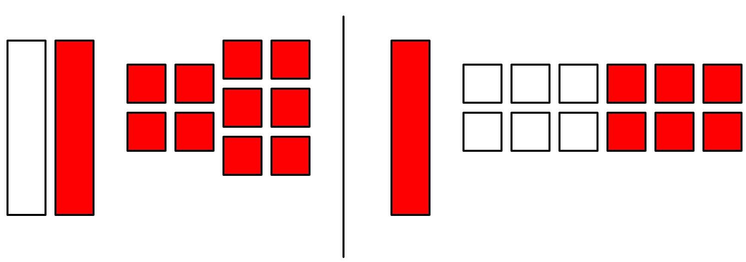
Step 3: Notice that what remains is $x = 2$.

Ex. 3: Solve $-x + 4 = -6$.

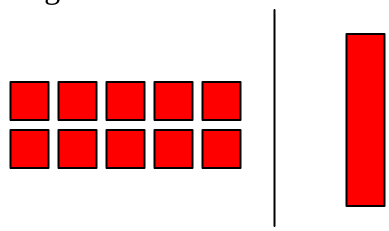
Step 1: Model $-x + 4$ on the left and -6 on the right.



Step 2: Add a $+x$ and six of the $+1$ to each side.



Step 3: Remove forms of zero, that is, the x and $-x$ on the left and the -6 and $+6$ on the right.



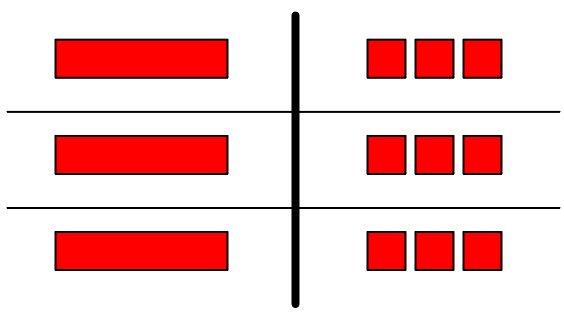
Step 3: Notice that what remains is $x = 10$.

When the coefficient of x is not 1, another two principles are useful to bring to bear.

The Sharing Principle

If two expressions are equivalent, you can divide each of them by the same number and maintain equivalence. Similarly, when two sets of tiles balance, if you can divide both sets into the same number of groups, each of which will balance.

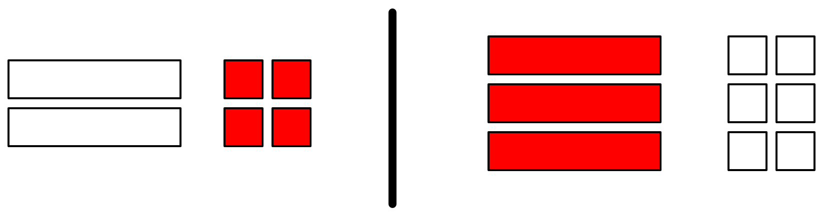
As an example, use algebra tiles to solve the equation $3x = 9$.



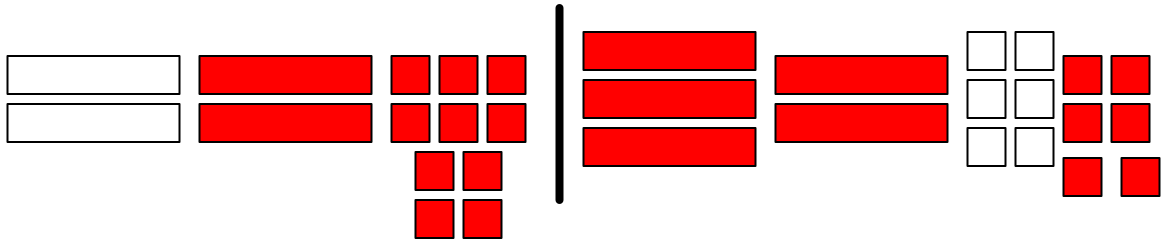
It is very easy to see that each side can be arranged into three equal groups showing that $x = 3$.

Similarly, to solve $-2x + 4 = 3x - 6$, do the following.

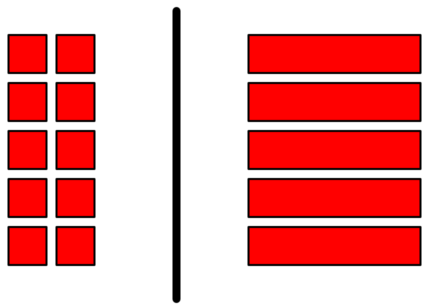
Step 1: Model $-2x + 4$ on the left and $3x - 6$ on the right.



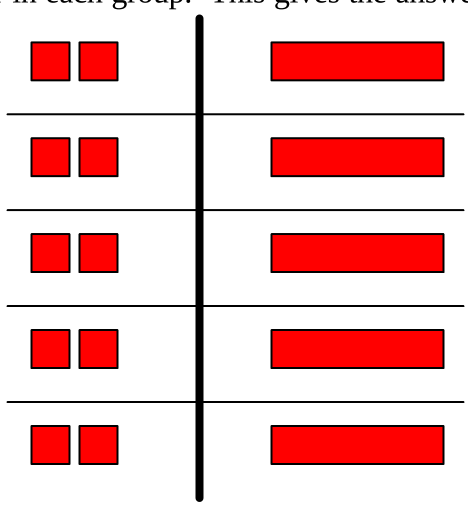
Step 2: Add 2 of the $+x$ and 6 of the $+1$ to each side.



Step 3: Remove forms of zero, that is, the $-2x$ and $2x$ on the left and the -6 and $+6$ on the right.



Step 4: Arrange the quantities on both sides into 5 equal groups, so that there is one x in each group. This gives the answer of $x = 2$.



Try These!

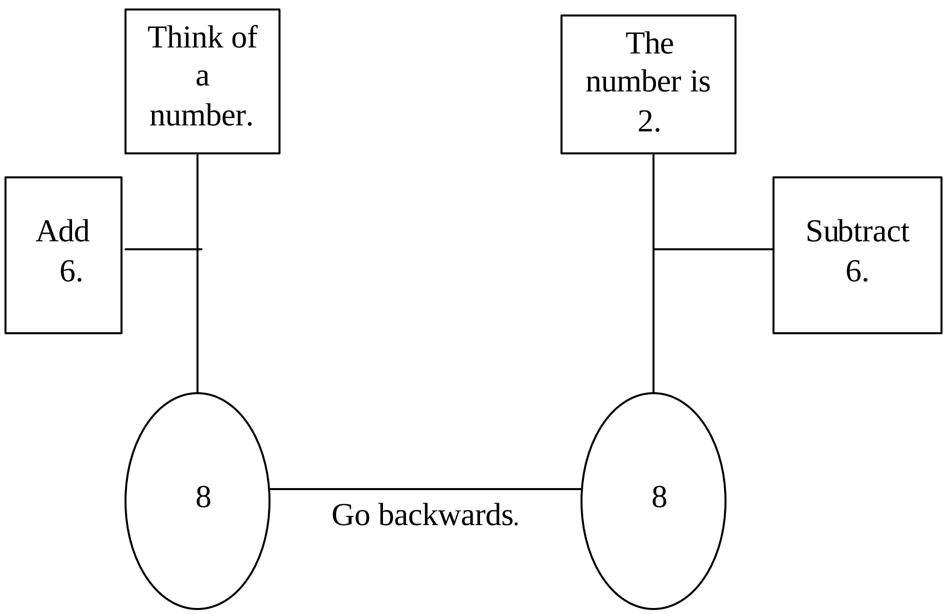
$x + 2 = 4$	$3x + 7 = 11$	$-7x = 24 - x$	$-12 + x = 15$
$c + 3 = -4$	$3c + 4 = -2$	$2c - 3 = -1$	$2t - 7 = 11$

Using Flow Charts

The same equations can be solved using a flow chart.

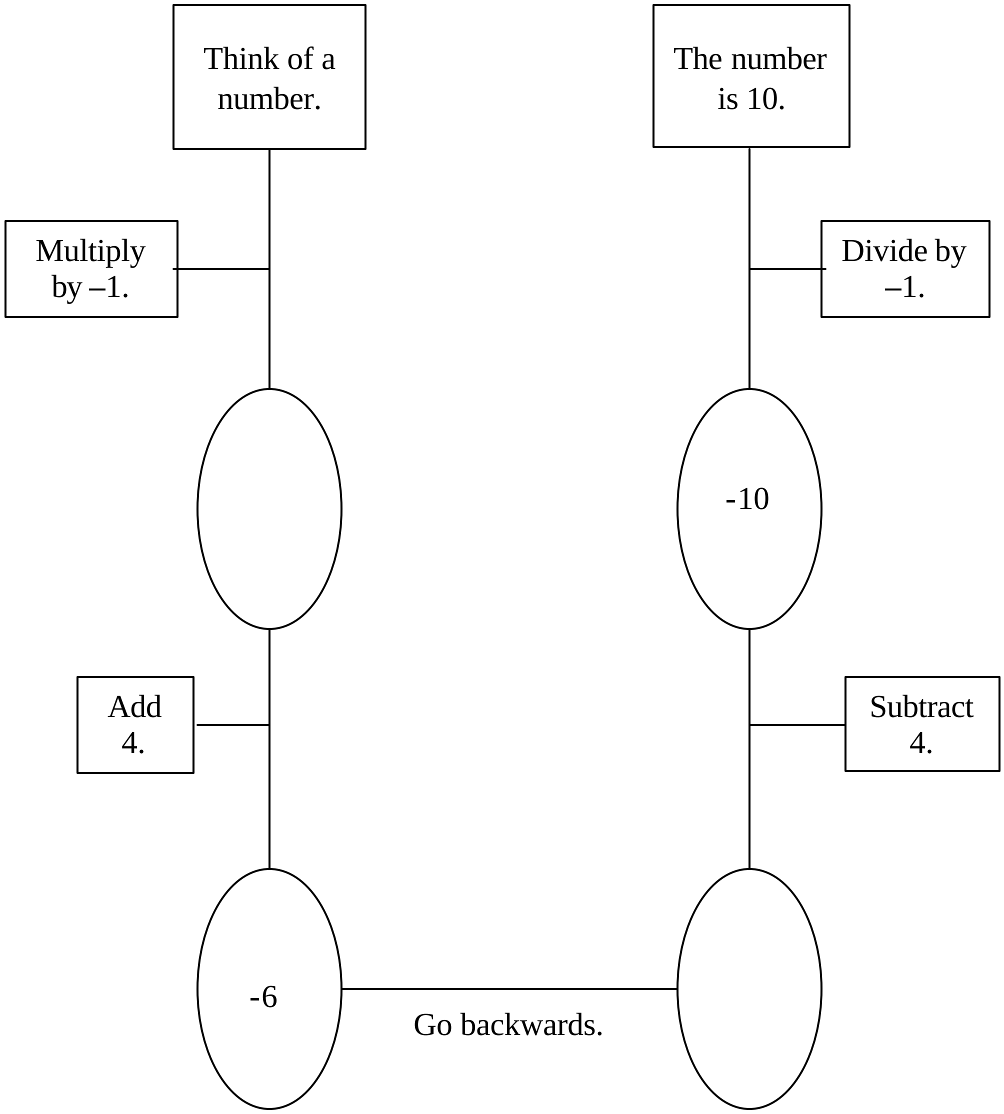
1. $x + 6 = 8$

   This means, think of a number and add 6. The result is 8. If you do everything in reverse, you should get back to the original number.



Since 8 – 6 = 2, the original number was 2.

2. $-x + 4 = -6$

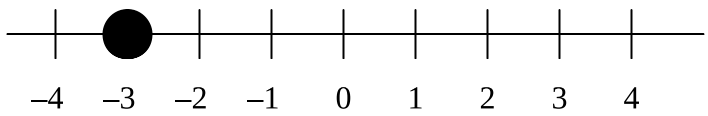


Since $-6 - 4 = -10$ and $-10 \div (-1) = 10$, the value of x is 10.

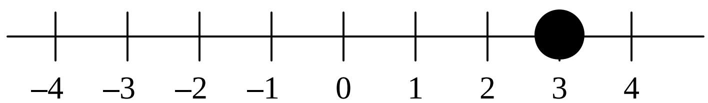
With certain equations, the flow chart method is not as helpful. For example, in the problem $-2x + 4 = 3x - 6$, determining how to work through the problem backwards is not as obvious.

GRAPHS OF EQUATIONS

A **graph** of an equation is a set of points that make the equation true. You might think of it as a picture of an equation. For example, the graph of $x = -3$ is pictured on the number line below.

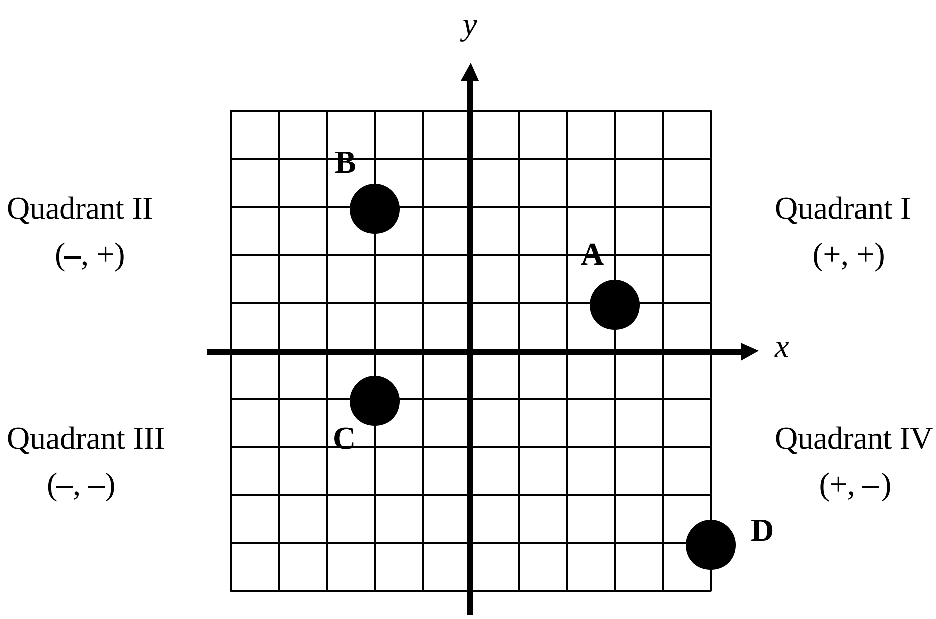


The graph of the equation $x = 3$ is shown below.



These are examples of plotting points in one dimension. You need only one axis, or number line. To plot points in two dimensions you need two number lines, one horizontal and the second vertical.

These two axes intersect at a point called the origin because it is the starting point for counting, representing the pair of numbers (0,0). You call the horizontal axis the x-axis and the vertical axis the y-axis. These two axes divide the graph paper into four regions, called quadrants.



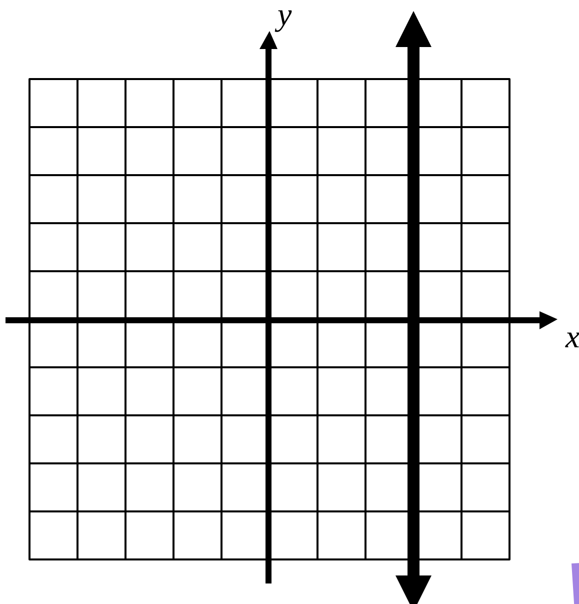
The first quadrant is the part above the *x*-axis and to the right of the *y*-axis. Every point in it represents a pair of positive numbers, such as (+3, +1) (point *A* on the graph).

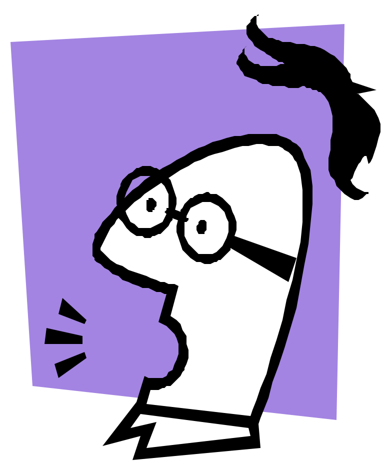
The second quadrant also lies above the *x*-axis, but to the left of the *y*-axis. Every point in it represents a pair of numbers, the first negative and the second positive; for example, (–2, +3) (point *B* on the graph).

The third quadrant lies to the left of the *y*-axis, but below the *x*-axis. Every point in it represents a pair of negative numbers, such as (–2, –1) (point *C* on the graph).

The fourth quadrant, like the first, lies to the right of the *y*-axis, but unlike the first lies below the *x*-axis. Every point in it represents a pair of numbers, the first a positive number and the second a negative number. The number pair (+5, –4) (point *D* on the graph) is an example.

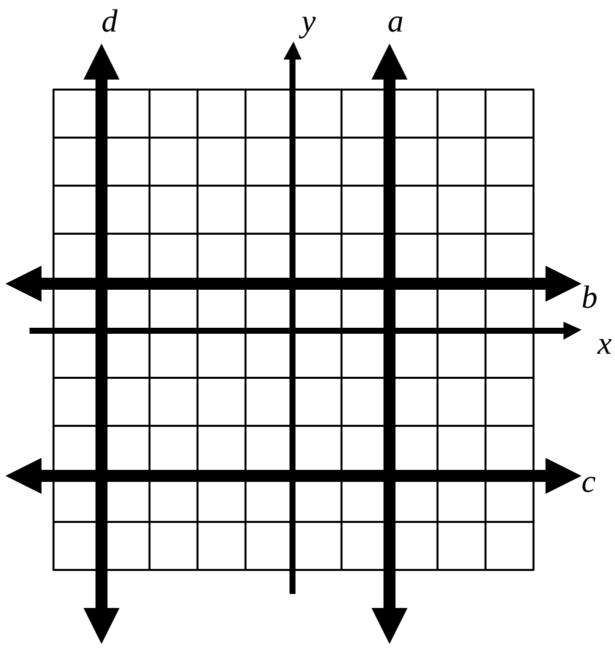
The equation $x = 3$ was first represented by a point on one number line. In a two-dimensional space, a second number line shows what points have 3 for their *x*–value.



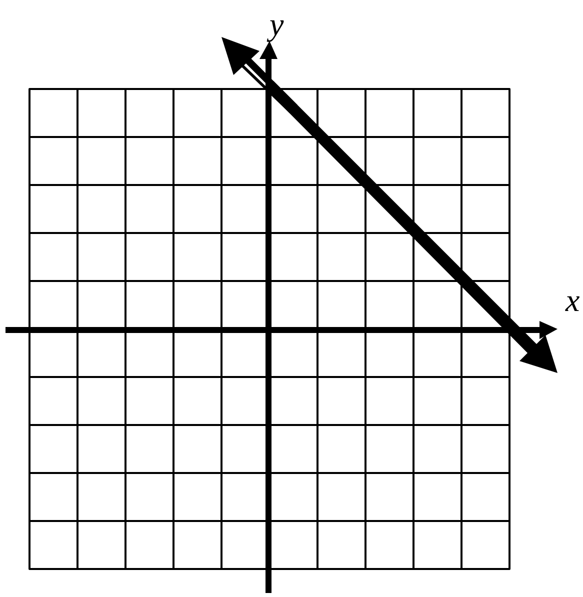


In the graph below:

Line *a* represents all the points where $x = 2$.
Line *b* represents all the points where $y = 1$.
Line *c* represents all the points where $y = -3$.
Line *d* represents all the points where $x = -4$.



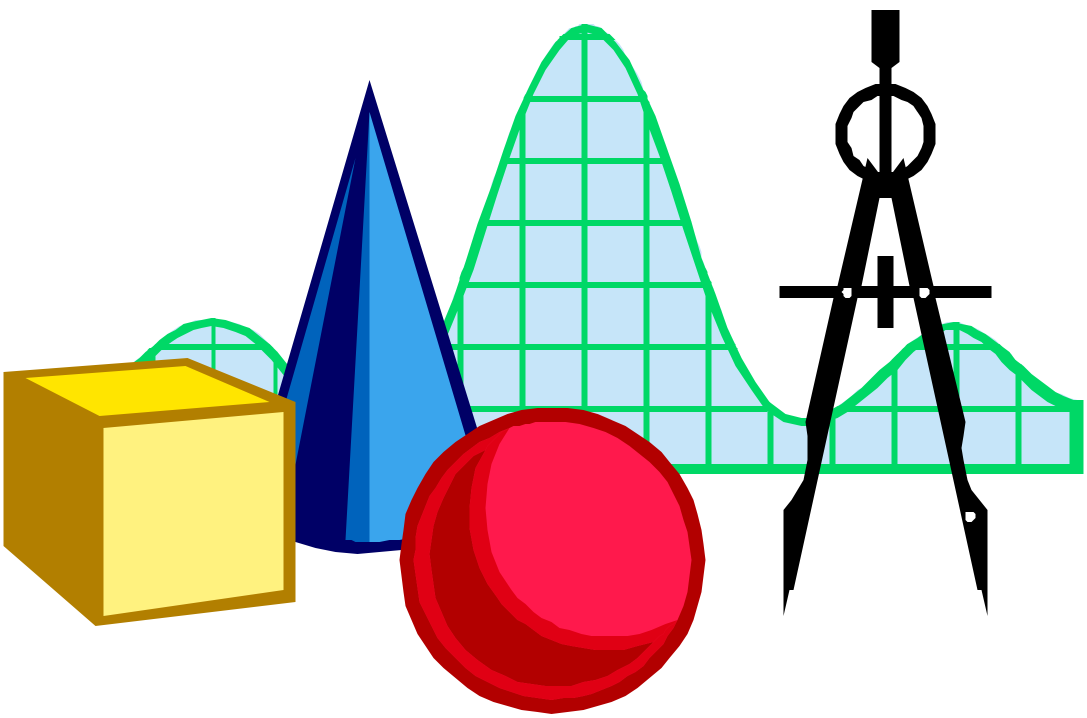
To show the equation $x + y = 5$, you plot the number pairs that will make this statement true. Some of these points are (2,3), (0,5), (5,0) and (1, –4). The coordinates of any point on the line will make the equation $x + y = 5$ true.



To graph a linear equation:

1. Solve the equation for y.
2. Choose two or more values for x.
3. Find the corresponding values for y.
4. Plot the points.
5. Join them in a straight line.

Use this technique to graph the equation of $2x + y = 7$.



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Greenes, Carole; Cavanagh, Mary; Dacey, Linda; Findell, Carol; and Small, Marian. *Navigating through Algebra in Prekindergarten-Grade 2*. Reston, Va: National Council of Teachers of Mathematics, 2001.